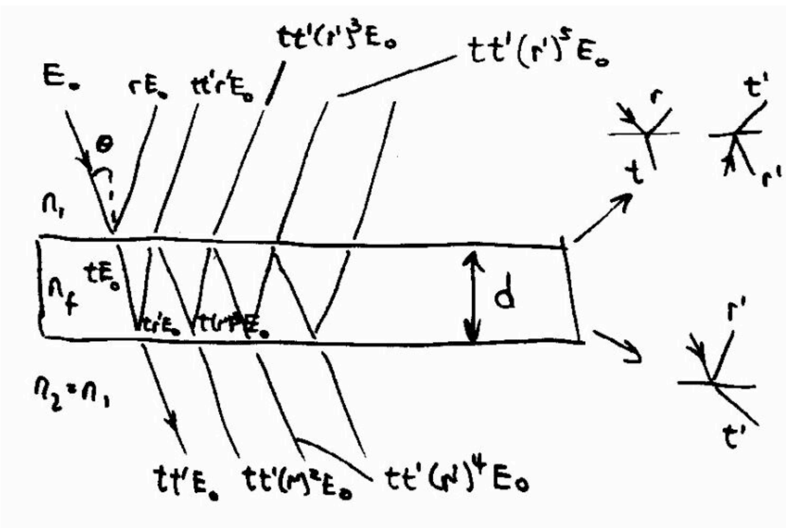
Multiple Beam Interference



consider relationship between r, r' and t, t':

$$r_{\parallel} = \frac{n_f\cos\theta - n_1\cos\theta_t}{n_1\cos\theta_t + n_f\cos\theta}, \quad r'_{\parallel} = \frac{n_1\cos\theta_t - n_f\cos\theta}{n_f\cos\theta + n_1\cos\theta_t} = -r_{\parallel}$$

$$t_{\parallel} = \frac{2n_1\cos\theta}{n_1\cos\theta_t + n_f\cos\theta}, \quad t'_{\parallel} = \frac{2n_f\cos\theta_t}{n_f\cos\theta + n_1\cos\theta_t}$$

$$T_{\parallel} \equiv \frac{n_f\cos\theta_t}{n_1\cos\theta}\, t_{\parallel}^2 = t'_{\parallel} t_{\parallel} \quad \text{and} \quad R_{\parallel} + T_{\parallel} = 1 = r_{\parallel}^2 + t_{\parallel}t'_{\parallel}$$

similarly $\quad r_{\perp} = -r'_{\perp} \quad$ and $\quad 1 = r_{\perp}^2 + t_{\perp}t'_{\perp}$

- use lenses to superimpose reflected beams, and transmitted beams,

between adjacent beams: $\Lambda = 2dn_f\cos\theta_t$

$\delta_R = \pi$ between top and bottom reflections (included in reflec. coeff, since $r = -r'$).

<u>special case</u>: $\Lambda = m\lambda_0$

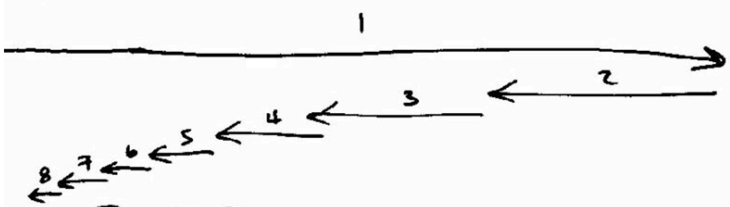
total: $E_{0r} = E_0 r - (E_0 tt'r + E_0 tt'r^3 + \dots)$

$$= E_0 r\left[1 - tt'\underbrace{(1 + r^2 + r^4 + \dots)}_{1/1-r^2}\right]$$

$$= E_0 r \left[1 - \frac{tt'}{1-r^2} \right]$$

$$= 0 \quad \text{since} \quad 1 - r^2 = tt'.$$

phasor diagram:



$\dots \quad E_{0r} = 0.$

<u>another special case</u>: $\Lambda = \left(m + \frac{1}{2}\right)\lambda_0$

now, 1st and 2nd beam in-phase; all other pairs (eg. 2nd, 3rd; 3rd, 4th; ...) out of phase.

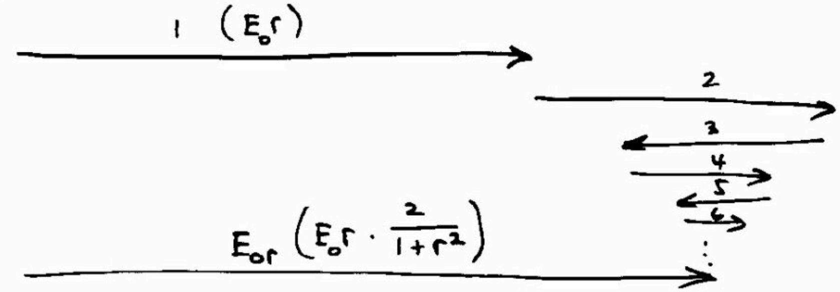
$$E_{0r} = E_0 r + E_0 tt'r - E_0 tt'r^3 + E_0 tt'r^5 - \dots$$

$$= E_0 r \left[1 + tt' \underbrace{\left(1 - r^2 + r^4 - \dots\right)}_{1/1+r^2} \right]$$

$$= E_0 r \left[1 + \frac{tt'}{1+r^2} \right]$$

$$= E_0 r \left[\frac{1 + r^2 + 1 - r^2}{1+r^2} \right] = E_0 \, \frac{2r}{1+r^2}$$

phasor diagram:



intensity $\quad I \propto \frac{1}{2} E_{0r}^2 = \frac{4r^2}{(1+r^2)^2} \left(\frac{E_0^2}{2} \right)$ $\quad$ max I_r, as will see.

<u>general case</u>:

use complex notation; review this:

say $\quad E_1 = E_{01} \cos(\alpha_1 + \omega t) = Re\Big(\underbrace{E_{01} e^{i(\alpha_1 + \omega t)}}_{\tilde{E}_1} \Big)$

add waves $\tilde{E}_1, \tilde{E}_2, \dots$

$$\tilde{E} = \left[\sum_{i=1}^{N} E_i e^{i\alpha_i} \right] e^{i\omega t} \qquad \equiv \tilde{E}_0 = E_0 e^{i\alpha} \quad \text{by phasor addition}$$

E_0 is <u>real</u>, so

$$I \propto \left\langle \left[Re(\tilde{E}) \right]^2 \right\rangle_T = \frac{1}{2} E_0^2 = \frac{1}{2} \tilde{E}_0 \tilde{E}_0^*$$

for reflections, write

$$\tilde{E}_{1r} = E_o r e^{i\omega t}$$

$$\tilde{E}_{2r} = -E_o t t' r e^{i(\omega t - \delta)}$$

$$\tilde{E}_{3r} = -E_o t t' r^3 e^{i(\omega t - 2\delta)}$$

$$\vdots$$

$$\tilde{E}_{Nr} = -E_o t t' r^{2N-3} e^{i(\omega t - (N-1)\delta)}$$

with
$\delta = k_o \Lambda$
$\Lambda = 2 n_f d \cos\theta_t$

then

$$\tilde{E}_r = E_o r e^{i\omega t}\left[1 - t t' e^{-i\delta}\underbrace{\left(1 + r^2 e^{-i\delta} + r^4 e^{-2i\delta} + \ldots\right)}_{1/1-r^2e^{-i\delta}}\right]$$

$$= E_o e^{i\omega t} r \left[\underbrace{1 - \frac{t t' e^{-i\delta}}{1 - r^2 e^{-i\delta}}}\right]$$

$$1 - \frac{(1-r^2)e^{-i\delta}}{1-r^2e^{-i\delta}} = \frac{1 - r^2 e^{-i\delta} - (1-r^2)e^{-i\delta}}{1 - r^2 e^{-i\delta}}$$

$$= E_o e^{i\omega t} r \left(\frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}}\right)$$

and

$$I_r \propto \frac{1}{2} E_o^2 r^2 \left(\frac{1-e^{-i\delta}}{1-r^2e^{-i\delta}}\right)\left(\frac{1-e^{i\delta}}{1-r^2e^{i\delta}}\right)$$

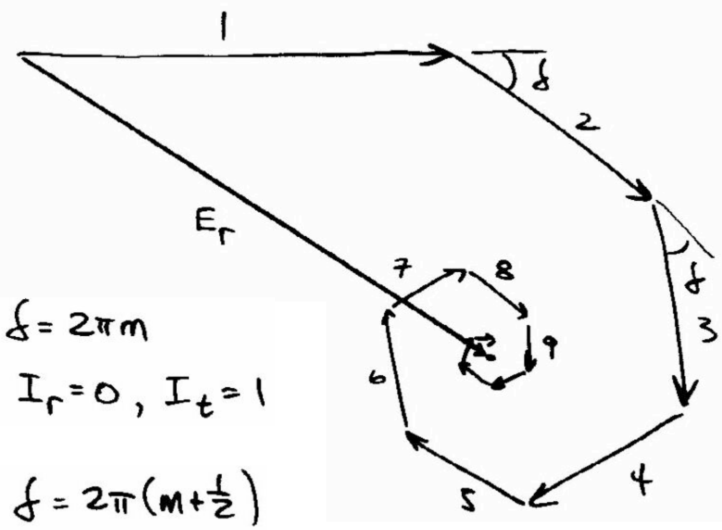
$$= \frac{1}{2}E_o^2 \frac{2r^2(1-\cos\delta)}{(1+r^4) - 2r^2\cos\delta}$$

similarly for transmitted wave, find

$$I_t = \frac{E_o^2}{2}\frac{(1-r^2)^2}{(1+r^4) - 2r^2\cos\delta}$$

and $I_t + I_r = \frac{E_o^2}{2} = I_i$ $\quad (R + T = 1$ ✓$)$

phasor diag:



min $\delta = 2\pi m$

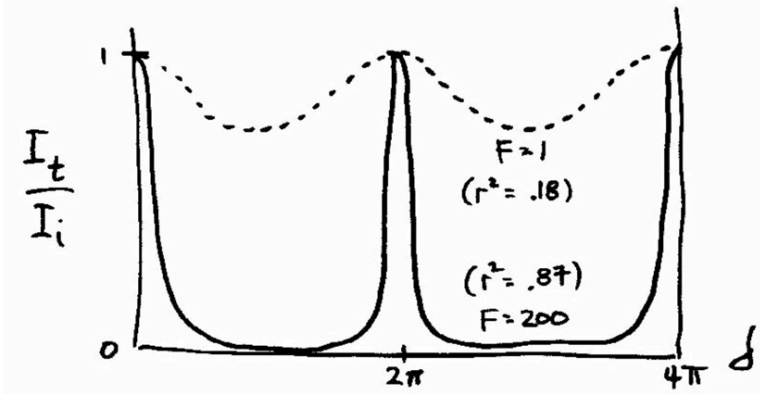
$I_r = 0, \; I_t = 1$

max $\delta = 2\pi(m + \frac{1}{2})$

$\cos\delta = -1$

$$I_r = I_i \cdot \frac{4r^2}{(1+r^4) + 2r^2} = I_i \cdot \frac{4r^2}{(1+r^2)^2}$$

$$I_t = I_i \cdot \frac{(1-r^2)^2}{(1+r^2)^2}$$



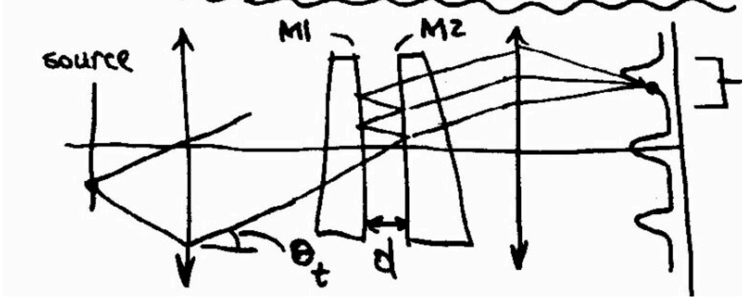
"coefficient of finesse" $\quad F \equiv \left(\frac{2r}{1-r^2}\right)^2$

so $\dfrac{I_r}{I_i} = \dfrac{F\sin^2(\delta/2)}{1+F\sin^2(\delta/2)}$, $\quad \dfrac{I_t}{I_i} = \dfrac{1}{1+F\sin^2(\frac{\delta}{2})}$

(using $\cos\delta = 1 - 2\sin^2(\frac{\delta}{2})$) $\qquad \equiv \mathcal{A}(\theta)$ Airy fcn.

Thus, multiple beam interference, for r large, yields sharp interference peaks.

Use in Fabry-Perot Interferometer



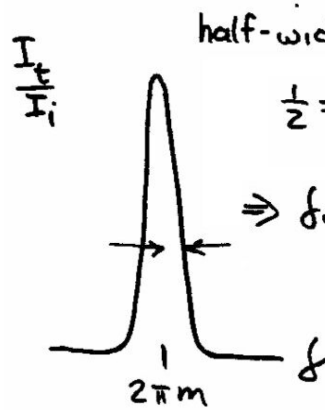
- use air film, $n_f = 1$ "etalon"
- usually have metal films on mirrors; give phase shift δ_R per reflection,

$$\delta = \frac{2\pi}{\lambda_0} 2d\cos\theta_t + 2\delta_R \qquad \text{between successive transmitted beams}$$

$\delta_R = \delta_R(\theta_t)$ but $\theta_t \approx 0$ so $\delta_R \approx$ const.

just introduces a shift in the δ axis, which is irrelevant.

In practice: achieve high spectral resolution ($r \to 1$) and vary d to probe fine spectral details (cf. Michelson Int. - only 2 beams!)



$\frac{1}{2} = \dfrac{1}{1+F\sin^2\frac{\delta_{1/2}}{2}}$

$\Rightarrow \delta_{1/2} = 2\sin^{-1}\left(\frac{1}{\sqrt{F}}\right) \approx \frac{2}{\sqrt{F}}$

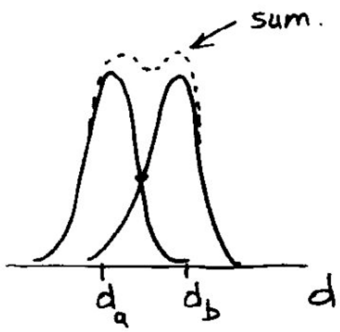
ratio of peak sep. to full width, "Finesse"

$$\mathcal{F} = \frac{2\pi}{4/\sqrt{F}} = \frac{\pi\sqrt{F}}{2}$$

want high $\mathcal{F}$!

say, observe source with two wavelengths:

λ_a, λ_b, same m



when are they resolved?

say, when

$$\Delta d \equiv d_b - d_a = 2 d_{1/2}$$

know $\delta_{1/2} = \frac{2}{\sqrt{F}}$ $\quad \delta = \frac{4\pi}{\lambda_o} d \cos\theta_t$, $\frac{\Delta\delta}{\Delta d} = \frac{4\pi}{\lambda_o}\cos\theta_t$

$$\Rightarrow d_{1/2} = \frac{2}{\sqrt{F}} \frac{\lambda_{AV}}{4\pi\cos\theta_t} \quad \text{with } \lambda_{AV} = \lambda_o.$$

$$\Delta d_{min} = 2 d_{1/2} = \frac{4}{\sqrt{F}} \cdot \frac{\lambda_{AV}}{4\pi} \frac{1}{\cos\theta_t}$$

express also in terms of $\Delta\lambda$:

$$\cancel{2\pi} m = \frac{\cancel{2\pi}}{\lambda} 2d\cos\theta_t$$

$$\Rightarrow d = \lambda \frac{m}{2\cos\theta_t} \Rightarrow \Delta d = \Delta\lambda \frac{m}{2\cos\theta_t}$$

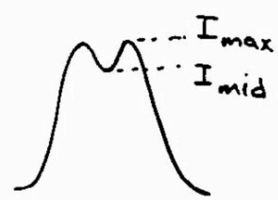
$$\Rightarrow \frac{4}{\sqrt{F}} \frac{\lambda_{AV}}{4\pi\cos\theta_t} = \Delta\lambda \frac{m}{2\cos\theta_t}$$

thus,

$$\underbrace{\frac{\lambda_{AV}}{\Delta\lambda_{min}} \equiv \mathcal{R}}_{\text{chromatic resolving power}} = \frac{\pi}{2} m\sqrt{F}$$

$$= \underline{\underline{m\mathcal{F}}}$$

(alternatively, use Raleigh criterion:



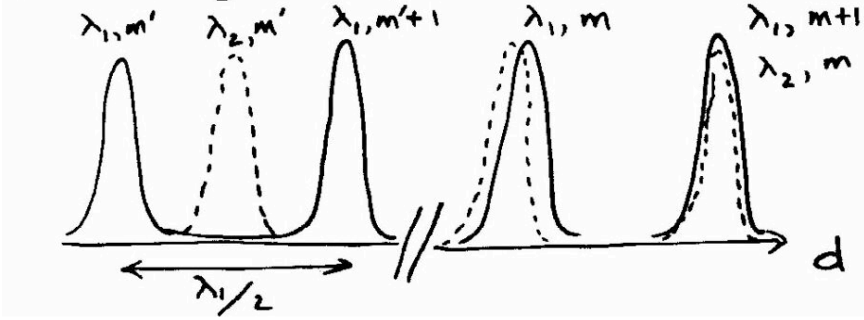
say

$$I_{mid} \le \frac{8}{\pi^2} I_{max}$$

leads to $\mathcal{R} = m\mathcal{F}(1.05)$).

trade off $\mathcal{R}$ with "free spectral range"

look when get overlap for different m:



$$\Lambda = 2d\,\cancel{\cos\theta_t}^{\,1} = m\lambda_0 \quad , \quad m = \frac{2d}{\lambda_0}$$

$m\lambda_2 = (m+1)\lambda_1$ write $\Delta\lambda = \lambda_2 - \lambda_1$

$\lambda_2 = \lambda_1 + \Delta\lambda$

$m(\Delta\lambda + \lambda_1) = m\lambda_1 + \lambda_1$

$$\Rightarrow \Delta\lambda = \frac{\lambda_1}{m} \; , \text{ or } \; (\Delta\lambda)_{fsr} = \frac{\lambda_0}{m} = \underline{\underline{\frac{\lambda_0^2}{2d}}}$$

adjust m (i.e. d) to trade off $\mathcal{R}$ and fsr.

Fourier Perspective (Michelson Int.)

2 beam int. $\delta = \frac{2\pi}{\lambda_0} 2d\,\cancel{\cos\theta_t} + \cancel{\delta_R}$

recall:

$$E^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos\delta$$

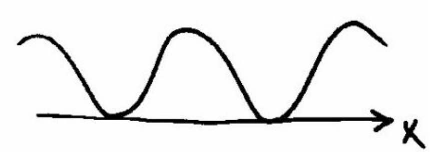
$$I = I_1^2 + I_2^2 + 2\sqrt{I_1}\sqrt{I_2}\cos\delta$$

$$\rightarrow 2I_0(1+\cos\delta) \quad \text{for } I_1 = I_2 \equiv I_0$$

$$= 2I_0(1+\cos kx) \quad \text{with } x \equiv 2d,\; k = k_0$$

$$\equiv I(k)$$

output:



input:

output is like F.T. of input spectrum!

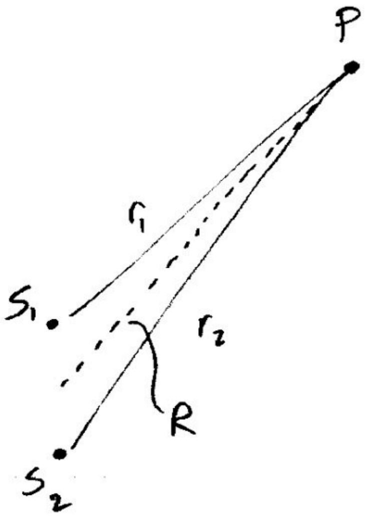
for more complicated input spectrum:

output:

$$I = \int_0^\infty I(k)\,dk = \underbrace{\int_0^\infty 2I_0(k)\,dk}_{\text{const.}} + \underbrace{\int_0^\infty 2I_0(k)\cos kx\,dx}_{2\tilde{I}(x)}$$

F.T.

back to interference problem:



at P,

$$E_1 = \frac{\mathcal{E}_1}{r_1} e^{i(kr_1 - \omega t)} \approx \frac{\mathcal{E}_1}{R} e^{i(kr_1 - \omega t)}$$

$$E_2 = \frac{\mathcal{E}_2}{r_2} e^{i(kr_2 - \omega t)} \approx \frac{\mathcal{E}_2}{R} e^{i(kr_2 - \omega t)}$$

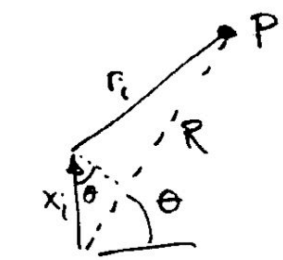
R large

$R = \frac{1}{2}(r_1 + r_2)$

total: $E = \sum_i E_i \approx \frac{1}{R} \sum_i \mathcal{E}_i e^{i(kr_i - \omega t)}$

$$= \frac{1}{R} e^{i(kR - \omega t)} \sum_i \mathcal{E}_i e^{ik\Delta r_i}$$

with $\Delta r_i = r_i - R$



($x_i < 0$ here)

P far away,

$\Delta r_i \approx -x_i \sin\theta$

thus,

$$E = \frac{1}{R} e^{i(kR - \omega t)} \sum_i \mathcal{E}_i e^{-ikx_i \sin\theta}$$

define "aperture fcn"

$$\mathcal{A}(x) = \sum_i \mathcal{E}_i \delta(x - x_i)$$

then Fourier transform is,

$$\tilde{\mathcal{A}}(q) = \int_{-\infty}^{\infty} e^{-iqx} \sum_i \mathcal{E}_i \delta(x - x_i)\, dx$$

$$= \sum_i \mathcal{E}_i e^{-iqx_i}$$

identify, $E_P \propto$ Fourier transform of $\mathcal{A}(x)$,

with $\boxed{q = k\sin\theta}$

and

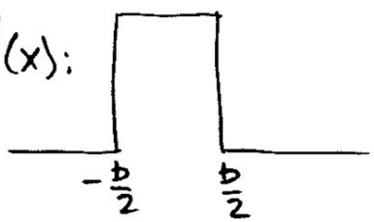
$$I \propto \langle E_P \rangle^2 \propto |A(q)|^2$$

"power spectrum"

Single Slit Diffraction

now consider wide source:

$A(x)$:



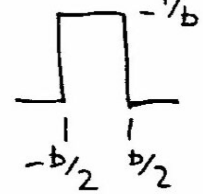
$$A(x) = \frac{1}{b}\left[\Theta\left(x+\frac{b}{2}\right) - \Theta\left(x-\frac{b}{2}\right)\right]\cdot \mathcal{E}_0$$

think about this as:

set of point sources (Huygen's prin.)

• single slit, finite width (Chap. 10)

$A(x)$:

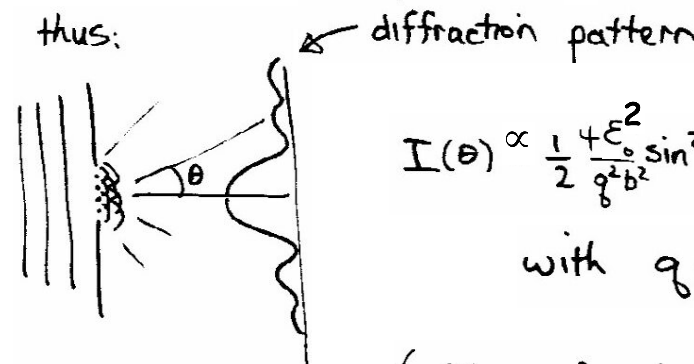


$$\tilde{A}(q) = \int_{-b/2}^{b/2} e^{iqx}\,dx \cdot \frac{\mathcal{E}_0}{b}$$

$$= \frac{\mathcal{E}_0}{iqb}\left(e^{iqb/2} - e^{-iqb/2}\right)$$

$$= \frac{2\mathcal{E}_0}{qb}\sin q\, b/2$$

thus:



$$I(\theta) \propto \frac{1}{2}\,\frac{4\mathcal{E}_0^2}{q^2 b^2}\sin^2 q\frac{b}{2}$$

with $q = k\sin\theta$

(valid even for large θ).

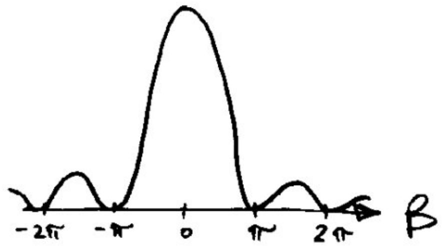
let $\beta \equiv \frac{qb}{2} = \frac{\pi b}{\lambda}\sin\theta$

$$\Rightarrow I(\theta) \propto \frac{\mathcal{E}_0^2}{\beta^2}\sin^2\beta = \mathcal{E}_0^2\left(\frac{\sin\beta}{\beta}\right)^2$$

examine single slit soln. in more detail:

$$I = I(0)\left(\frac{\sin\beta}{\beta}\right)^2$$

$$\beta = \frac{qb}{2} = \frac{\pi}{\lambda} b\sin\theta$$

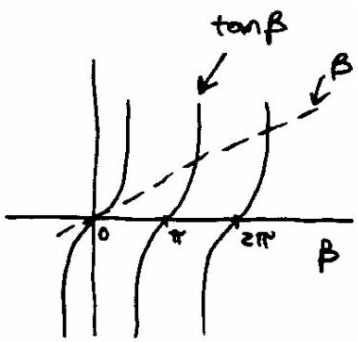


zeroes at $\beta = m\pi$, $m \neq 0$

maxima at:

$$\frac{dI}{d\beta} = I(0)\, 2\, \frac{\sin\beta}{\beta}\underbrace{\left(\frac{\beta\cos\beta - \sin\beta}{\beta^2}\right)} = 0$$

$\Rightarrow \beta\cos\beta - \sin\beta = 0$, $\underline{\tan\beta = \beta}$



$\Rightarrow \beta = 0,$
$\pm 1.43\pi,$
$\pm 2.46\pi,$
$\vdots$
$\pm(n+\frac{1}{2})\pi$

$$\left(\text{also, } \frac{dI}{d\beta}(0) \to \left.\frac{\cos\beta - \beta\sin\beta - \cos\beta}{2\beta}\right|_{\beta=0} \to 0 \right)$$

consider: <u>an array of finite slits:</u>

S(x):

N slits

• can we derive this from previous result for single finite slit and N δ-fcn slits?

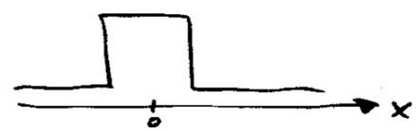
<u>YES</u> - use convolution (chap. 11)

Thm: if $g(x) = \int_{-\infty}^{\infty} dx'\, f(x')\, h(x-x')$

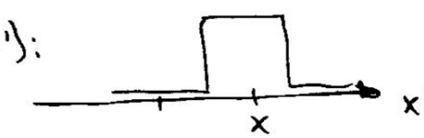
then $\tilde{g}(q) = \tilde{f}(q) \cdot \tilde{h}(q)$ Fourier transforms

<u>example</u>:

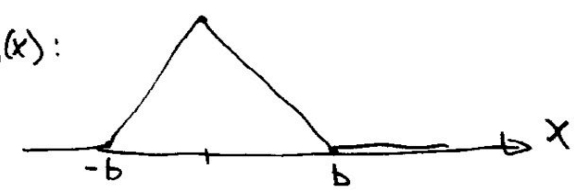
$f(x')$:



$h(x-x')$:



$g(x)$:



have

$$A(x) = \sum_{i=1}^{N} \left[\theta\left(x - x_i + \frac{b}{2}\right) - \theta\left(x - x_i - \frac{b}{2}\right) \right] \frac{1}{b}\, \mathcal{E}_0$$

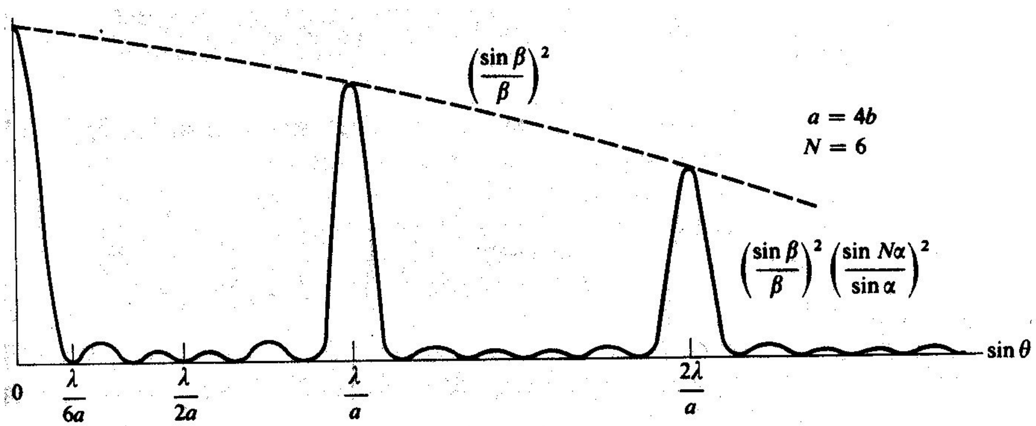
$$= \int_{-\infty}^{\infty} dx' \underbrace{\sum_{i=1}^{N} \delta(x - x_i - x')}_{h(x-x')} \underbrace{\left[\theta\left(x' + \frac{b}{2}\right) - \theta\left(x' - \frac{b}{2}\right) \right]}_{f(x')} \frac{\mathcal{E}_0}{b}$$

$$\tilde{f}(q) = \frac{\mathcal{E}_0}{\beta} \sin\beta \,, \quad \beta = \frac{qb}{2}$$

$$h(x) = \sum_{i=1}^{N} \delta(x - x_i)$$

$$\tilde{h}(x) = \frac{\sin N\alpha}{\sin\alpha} \quad \text{with} \quad \alpha = \frac{qa}{2}$$

$$\Rightarrow I(\theta) \propto \bar{\mathcal{E}}_0^2 \left(\frac{\sin N\alpha}{\alpha}\right)^2 \cdot \left(\frac{\sin\beta}{\beta}\right)^2$$



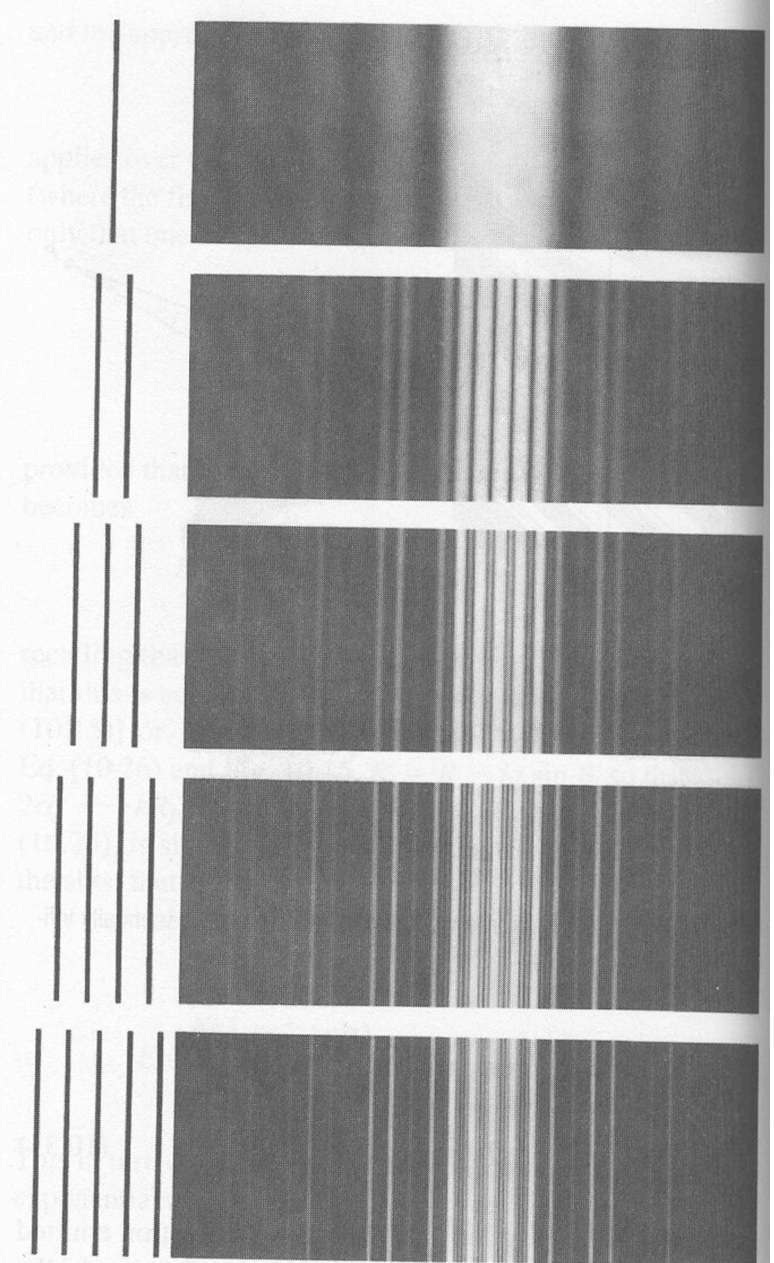
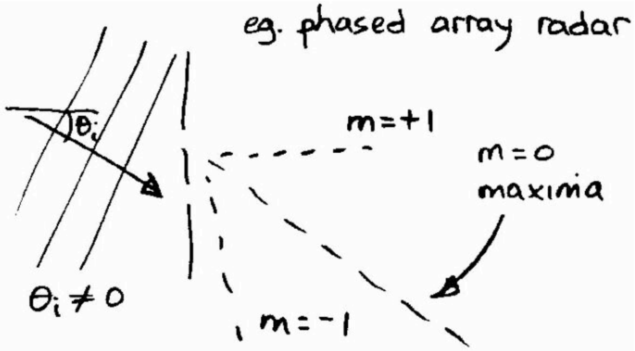


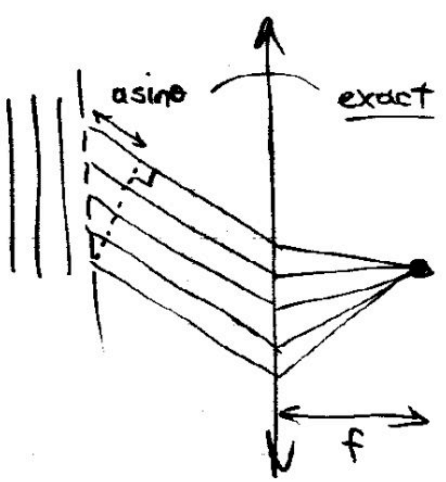
Figure 10.16 Diffraction patterns for slit systems shown at left. (Francis Weston Sears, *Optics*. Reprinted with permission of Addison Wesley Longman, Inc.)

note: for point (i), can have angle on incident wave:



(this requires small modification of F.T. approach using aperture fcn; replace exit angle $\sin\theta$ by $\sin\theta - \sin\theta_i$).

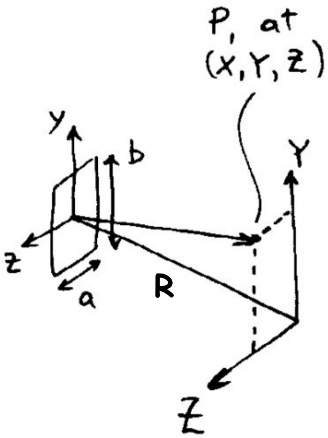
also,



always in far-field.

2-D Apertures

(a) rectangular



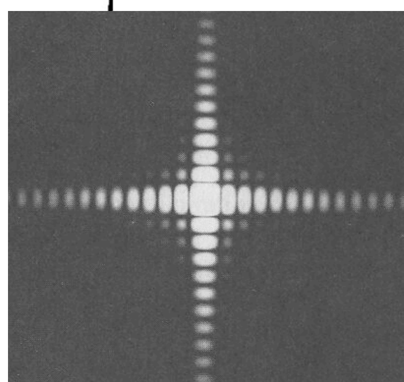
$$I(Y,Z) \propto \left|\tilde{A}(q_Y, q_Z)\right|^2$$

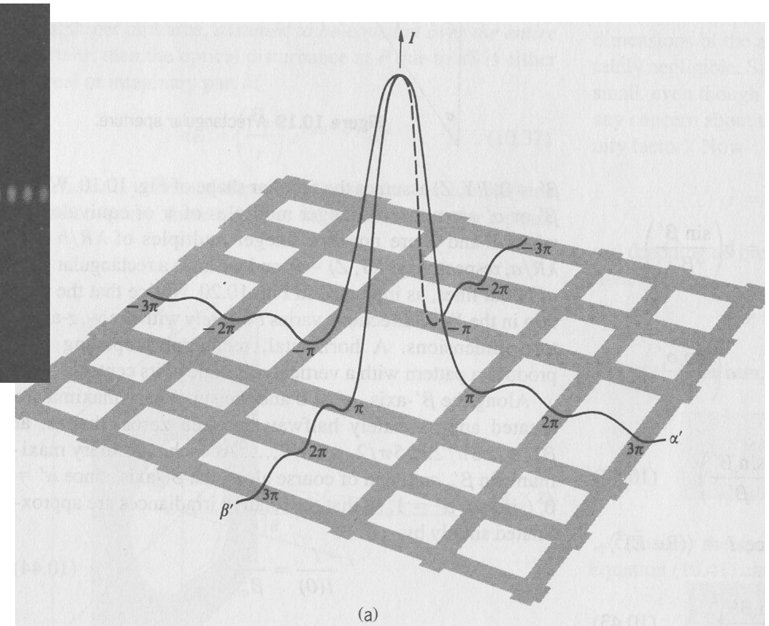
$$q_Y = k\sin\theta_Y \approx \frac{kY}{R}, \quad q_Z = k\sin\theta_Z \approx \frac{kZ}{R}$$

$$\tilde{A} = \int_{-a/2}^{a/2} dz \int_{-b/2}^{b/2} dy\, e^{-i(q_Y y + q_Z z)}$$

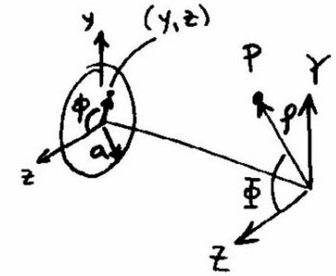
$$= \frac{2}{q_Y b} \sin q_Y \frac{b}{2} \cdot \frac{2}{q_Z a} \sin q_Z \frac{a}{2}$$

$$\Rightarrow I(Y,Z) = I(0) \left(\frac{\sin\alpha'}{\alpha'}\right)^2 \left(\frac{\sin\beta'}{\beta'}\right)^2 \text{ with } \alpha' = \frac{kYb}{2R}, \ \beta' = \frac{kZa}{2R}$$





(b) circular



$$\tilde{A} = \iint_{\text{circle}} dy\,dz\; e^{\overbrace{-i(q_Y y + q_Z z)}}$$

$\vec{q}\cdot\vec{r}$ with $\vec{q} = (q_Y, q_Z)$

$\vec{r} = (y, z)$

$$q = (q_Y^2 + q_Z^2)^{1/2} = k\sin\theta \approx \frac{k\rho}{R}$$

with $\rho = (Y^2 + Z^2)^{1/2}$

$$\vec{q}\cdot\vec{r} = qr\cos(\Phi - \phi)$$

$$\tilde{A} = \int_0^{2\pi} d\phi \int_0^a r\,dr\; e^{-iqr\cos(\Phi-\phi)}$$

$$= \int_0^a r\,dr \underbrace{\int_0^{2\pi} e^{-iqr\cos\phi}\,d\phi}_{2\pi J_0(qr)}$$

so, $\tilde{A} = \int_0^a r\,dr\; 2\pi J_0(qr)$

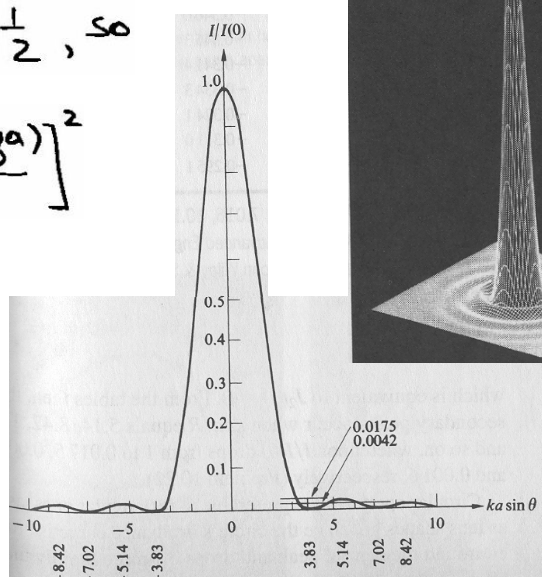
$$= 2\pi a^2 \frac{1}{qa} J_1(qa)$$

also, $\lim_{x\to 0} \frac{J_1(x)}{x} = \frac{1}{2}$, so

$$I = I(0)\left[\frac{2J_1(qa)}{qa}\right]^2$$

zeroes of $J_1(x)$:

$x = 0,$
$3.83,$
$7.02,$
$10.17,$
$\ldots\ (n + 1/4)\pi$



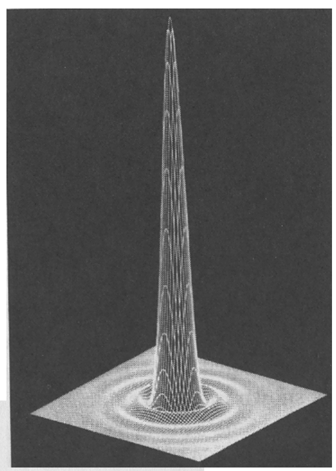
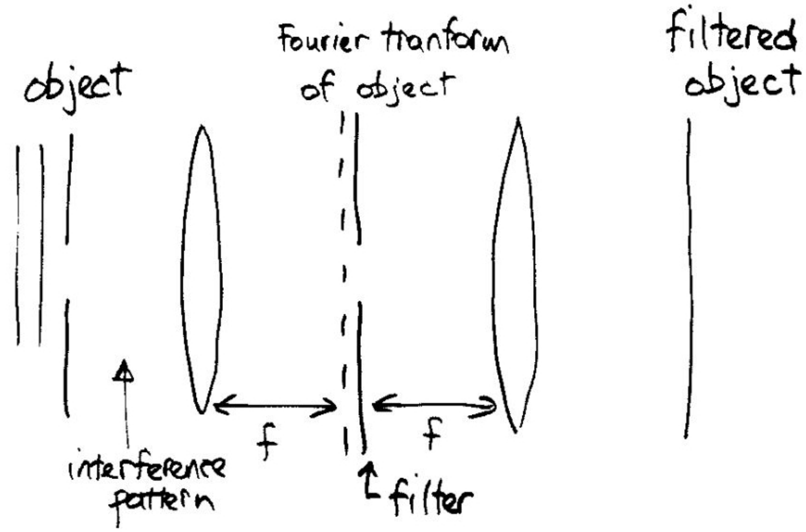


Image Analysis

use optics (far-field diffraction) to form Fourier transform, and then filter and invert that.



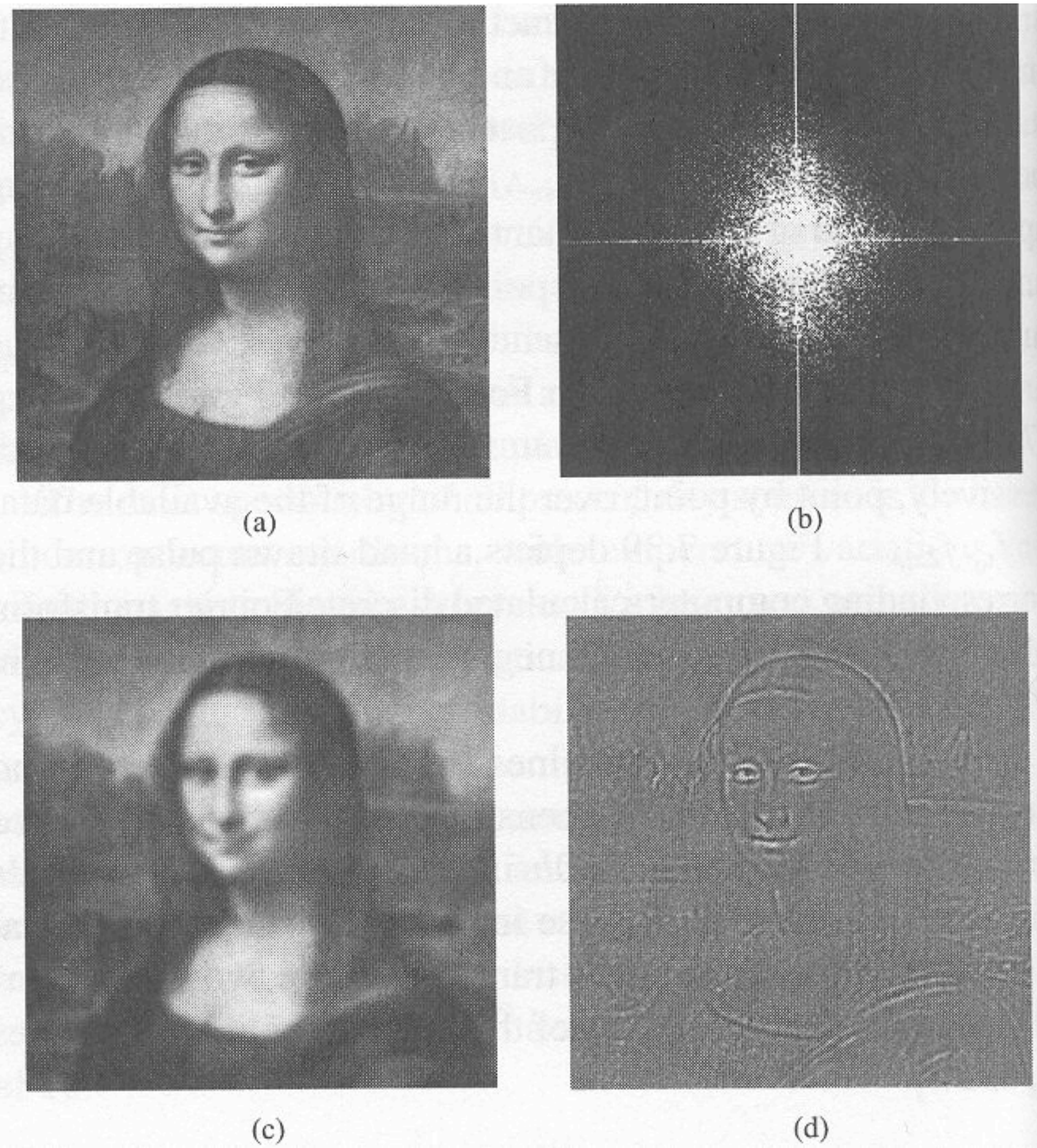


Figure 7.41 (*a*) The Mona Lisa and (*b*) the central portion of its power spectrum. (*c*) Mona, with her high spatial frequencies removed. (*d*) Mona with her low spatial frequencies removed. (Photos courtesy Synoptics Image Processing Systems, Cambridge, UK.)